

ON CURVILINEAR CONGRUENCES*

BY

C. E. WEATHERBURN

In a recent paper *On congruences of curves*† the writer has shown how the theory of such congruences in ordinary space of three dimensions may be extended along the lines followed for rectilinear congruences, making use of oblique curvilinear coördinates. The present paper contains a further extension of the theory, but is mainly independent of the preceding investigation. The subject is approached from a slightly different point of view, with the aid of the differential invariants grad, div, rot for three dimensions and, as far as possible, the use of coördinates is avoided. Some of the theorems proved have hitherto been known only for rectilinear congruences, while others again lose their significance in the particular case in which the lines are straight.

1. FIRST QUADRIC. CONE OF ZERO TENDENCY

Given a congruence of curves, the unit vector t tangent to the curve at any point P is known as a point function in space. It has a definite derivative for each direction. In the direction of the unit vector a its derivative‡ is $a \cdot \nabla t$, and the resolved part of this derivative in the direction of a has the value $a \cdot \nabla t \cdot a$, the operator ∇ being understood to act only on the vector immediately following it. The quantity just defined may be called the *tendency* of the congruence at P in the direction of a . It plays an important part in the following argument. Denoting it by T we have

$$(1) \quad T = a \cdot \nabla t \cdot a.$$

If then we introduce the quadric§

$$(2) \quad r \cdot \nabla t \cdot r = 1$$

with a center at the point P , which is origin for the vector r , it is clear that *the value of T for any direction at P is equal to the inverse square of the radius of the quadric (2) in that direction.* This square may be either positive or nega-

* Presented to the Society, October 29, 1927; received by the editors in September, 1927.

† *On congruences of curves*, Tôhoku Mathematical Journal, vol. 28 (1927), p. 114.

‡ Cf. the author's *Advanced Vector Analysis*, Art. 6.

§ Ibid., Art. 66.

tive. Also from the definition of the "divergence"* of a vector it follows that the sum of the tendencies for three mutually perpendicular directions at a point is invariant, and equal to $\text{div } \mathbf{t}$. Hence:

The sum of the tendencies of the congruence in three mutually perpendicular directions at a point is invariant, and equal to the divergence of the congruence at that point.

The asymptotic cone of the quadric (2) is given by

$$(3) \quad \mathbf{r} \cdot \nabla \mathbf{t} \cdot \mathbf{r} = 0.$$

This may be called the *cone of zero tendency* at P , for in the direction of any of its generators, the tendency of the congruence is zero. The tangent to the curve at P is clearly a generator of this cone; for, since $\mathbf{t}$ is a unit vector, its derivative in the direction of the tangent is perpendicular to $\mathbf{t}$.

We shall be concerned largely with the section of the quadric (2) by the normal plane of the curve at P , that is to say, by the plane perpendicular to $\mathbf{t}$. This section is the conic

$$(4) \quad \mathbf{r} \cdot \mathbf{t} = 0, \quad \mathbf{r} \cdot \nabla \mathbf{t} \cdot \mathbf{r} = 1$$

whose asymptotes are the corresponding section of the cone (3), giving the directions of zero tendency in the normal plane. For a direction inclined at an angle θ in the normal plane to that of a principal axis of the conic (4), the tendency is given by

$$T = T_1 \cos^2 \theta + T_2 \sin^2 \theta$$

where T_1 is the tendency in the direction of the above axis, and T_2 that for the perpendicular direction. Obviously $T_1 + T_2 = \text{div } \mathbf{t}$, since the tendency in the direction of $\mathbf{t}$ is zero.

We may observe that a direction of zero tendency in the normal plane corresponds to that of a "common perpendicular" to nearby rays in the case of a rectilinear congruence.† This should be borne in mind in order to see the analogy which the following theorems present to those already known for a congruence of straight lines.

2. SURFACE OF STRICTION. LIMIT SURFACE

The two directions of zero tendency in the normal plane will be at right angles provided $\text{div } \mathbf{t}$ is zero. This follows from the above theorem on the

* *Advanced Vector Analysis*, Art. 7.

† Cf. the author's *Differential Geometry*, Chap. X.

invariance of the sum of the tendencies in three perpendicular directions. The locus of the points at which this relation holds may be called the middle surface or *surface of striction* of the congruence, being analogous to the line of striction of a family of curves on a surface.* The points in which it is cut by any curve are the *points of striction* of the curve. Thus:

The locus of points at which the two directions of zero tendency in the normal plane are at right angles is the surface of striction, or middle surface, and is given by $\text{div } \mathbf{t} = 0$.

For a congruence of straight lines the middle surface† is the locus of points midway between the limits. In the case of a normal congruence the two directions of zero tendency in the normal plane are the asymptotic directions for the surface orthogonal to the curves. These directions are perpendicular when the mean curvature is zero. At such points $\text{div } \mathbf{t} = 0$ in agreement with the above.

The *limit surface*‡ may be defined as the locus of points at which the two directions of zero tendency in the normal plane are coincident. At such points the normal plane touches the cone of zero tendency. Now the normals to the quadric cone $\mathbf{r} \cdot \nabla \mathbf{t} \cdot \mathbf{r} = 0$ at its vertex generate another, called its reciprocal cone, whose equation is

$$\mathbf{r} \cdot (\nabla \mathbf{t})^{-1} \cdot \mathbf{r} = 0$$

where $(\nabla \mathbf{t})^{-1}$ is the reciprocal dyadic to $\nabla \mathbf{t}$. Now the "second"§ of $\nabla \mathbf{t}$, which is denoted by $(\nabla \mathbf{t})_2$, is proportional to the conjugate of $(\nabla \mathbf{t})^{-1}$. Consequently the equation of the reciprocal cone may also be expressed as

$$(5) \quad \mathbf{r} \cdot (\nabla \mathbf{t})_2 \cdot \mathbf{r} = 0.$$

Thus, at points on the limit surface, the tangent to the curve must be a generator of the cone (5).

Let i, j, k be three fixed perpendicular unit vectors forming a right-handed system. Let i be parallel to $\mathbf{t}$, and the other two therefore parallel to the normal plane of the curve at the point considered. Also let t_1, t_2, t_3 denote the derivatives of $\mathbf{t}$ in these three directions. Then

$$\nabla \mathbf{t} = i t_1 + j t_2 + k t_3,$$

and

$$(\nabla \mathbf{t})_2 = i t_2 \times t_3 + j t_3 \times t_1 + k t_1 \times t_2.$$

* *Differential Geometry*, Art. 126.

† *On congruences of curves*, Art 9.

‡ *Ibid.*, Art. 4.

§ E. B. Wilson, *Vector Analysis*, pp. 316-317.

Hence the tangent to the curve will be a generator of the cone (5) provided $\mathbf{t} \cdot (\nabla \mathbf{t})_2 \cdot \mathbf{t} = 0$, that is

$$(6) \quad \mathbf{t} \cdot (\mathbf{t}_2 \times \mathbf{t}_3) = 0.$$

Again, with the same notation, we have

$$\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t} = \mathbf{i} \times (\mathbf{t} \times \mathbf{t}_1) + \mathbf{j} \times (\mathbf{t} \times \mathbf{t}_2) + \mathbf{k} \times (\mathbf{t} \times \mathbf{t}_3)$$

from which it is easily verified that, $\mathbf{t}$ being a unit vector,

$$\operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) = 2 \mathbf{i} \cdot (\mathbf{t}_2 \times \mathbf{t}_3) = 2 \mathbf{t} \cdot (\mathbf{t}_2 \times \mathbf{t}_3).$$

Thus the condition (6) is equivalent to

$$\operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) = 0.$$

Hence the theorem:

The locus of points at which the two directions of zero tendency in the normal plane are coincident is the limit surface, whose equation may be expressed as

$$(7) \quad \operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) = 0,$$

or

$$(7') \quad \operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} + \mathbf{t} \times \operatorname{rot} \mathbf{t}) = 0.$$

In the case of a normal congruence of curves $\mathbf{t}$ is the unit normal to the surface orthogonal to the curves. Then, since the first member of (7) is twice the Gaussian curvature of the surface,* it follows that

The limit surface of a normal congruence is the locus of points at which the Gaussian curvature of the orthogonal surfaces is zero.

This agrees with the known property of a surface that the asymptotic directions are coincident where the Gaussian curvature vanishes.

3. SECOND QUADRIC. CONE OF ZERO MOMENT

Again, let $\mathbf{t}$ be the unit tangent at P , and $\mathbf{t} + \delta \mathbf{t}$ that at a nearby point Q , such that the vector PQ is $\delta s \mathbf{a}$, δs being the length of PQ and $\mathbf{a}$ a unit vector. The mutual moment of the tangents at P and Q , being the resolved part in the direction of $\mathbf{t}$ of the moment of $\mathbf{t} + \delta \mathbf{t}$ about P , has the value

$$\delta s \mathbf{a} \times (\mathbf{t} + \delta \mathbf{t}) \cdot \mathbf{t} = \delta s (\mathbf{a} \times \delta \mathbf{t}) \cdot \mathbf{t}.$$

The quotient of this mutual moment by $(\delta s)^2$ has the value $\mathbf{a} \times (\delta \mathbf{t} / \delta s) \cdot \mathbf{t}$,

* *Differential Geometry*, Art. 131.

and the limit of this as the point Q tends to coincidence with P , while the direction of $\mathbf{a}$ remains constant, is the function

$$(8) \quad M = \mathbf{a} \times (\mathbf{a} \cdot \nabla \mathbf{t}) \cdot \mathbf{t} = \mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a}.$$

We shall call this the *moment* of the congruence at P for the direction of $\mathbf{a}$. Let us now introduce the quadric

$$(9) \quad \mathbf{r} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{r} = 1$$

with a center at P , which is origin for the vector $\mathbf{r}$. Then it is clear from (8) that

The moment of the congruence for any direction at P is equal to the inverse square of the radius of the quadric (9) in that direction, having the value zero for directions in the asymptotic cone

$$(10) \quad \mathbf{r} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{r} = 0.$$

This cone of zero moment was known to Malus, and has been called by Darboux* the *cone of Malus*. It was found by investigating the directions at P which give nearby points, such that the tangent to the curve at one of these points and the tangent at P have a shortest distance apart of the second or higher order. It is the counterpart of the cone (3) of zero tendency. The tangent at P is clearly a generator of the cone of Malus.

Again it follows from (8) and (9) that the sum of the moments of the congruence in three mutually perpendicular directions at P is invariant. The value of this sum is the "scalar"† of $\nabla \mathbf{t} \times \mathbf{t}$, which is easily shown to have the value $\mathbf{t} \cdot \text{rot } \mathbf{t}$. Thus:

The sum of the moments of the congruence for any three mutually perpendicular directions at a point is invariant and equal to $\mathbf{t} \cdot \text{rot } \mathbf{t}$.

This quantity $\mathbf{t} \cdot \text{rot } \mathbf{t}$ may therefore be called the *total moment* of the congruence at P . It vanishes when the congruence is normal. For, in this case, $\mathbf{t}$ is the unit normal to the surface of a singly infinite family, and may therefore be expressed in the form $\psi \nabla \phi$, where ψ and ϕ are point functions. Consequently $\text{rot } \mathbf{t} = \nabla \psi \times \nabla \phi$, and

$$\mathbf{t} \cdot \text{rot } \mathbf{t} = \psi \nabla \phi \cdot (\nabla \psi \times \nabla \phi) = 0.$$

In this case the cone of Malus has an infinite number of sets of three mutually perpendicular generators.‡

* *Leçons sur la Théorie Générale des Surfaces*, vol. 2, pp. 262, 280.

† *Advanced Vector Analysis*, Art. 56.

‡ Darboux, loc. cit.

We shall be concerned largely with the sections of the quadric (9) and the cone of Malus by the normal plane at P . The section of the former is the conic

$$(11) \quad \mathbf{r} \cdot \mathbf{t} = 0, \quad \mathbf{r} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{r} = 1$$

whose asymptotes are the directions of zero moment in the normal plane. For a direction inclined at an angle θ in this plane to a principal axis of the conic (11), the moment is given by

$$(12) \quad M = M_1 \cos^2 \theta + M_2 \sin^2 \theta,$$

where M_1 is the moment in the direction of the above axis, and M_2 is that for the perpendicular direction. And clearly $M_1 + M_2 = \mathbf{t} \cdot \text{rot } \mathbf{t}$, since the direction of $\mathbf{t}$ is one of zero moment. It should also be observed that the moment in any direction may be negative. This is the case when the point of the quadric in that direction is imaginary.

We may here pause to draw attention to an interesting result in connection with rectilinear congruences. We have shown elsewhere that, on a ruled surface, the moment M of the family of generators and the Gaussian curvature K at that point are connected by the* relation $M = \pm (-K)^{1/2}$. Consider then any surface of the rectilinear congruence. Let $\mathbf{a}$ be the unit vector which is normal to the ray and tangential to the surface. Then the moment in this direction is $\mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a}$, and the Gaussian curvature of the ruled surface at P is given by

$$K = - [\mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a}]^2.$$

This vanishes only when $\mathbf{a}$ is a direction of zero moment; and if this is so for every point P , the surface is a developable surface of the congruence.

4. SURFACE OF NORMALITY. ULTIMATE SURFACE

If the two directions of zero moment in the normal plane are at right angles, the total moment at that point is zero; and conversely. The locus of points at which the congruence possesses this property may be called the *surface of normality*, for at such points the condition is satisfied that the congruence should be normal. It is the surface of zero total moment, and the points in which it is cut by any curve are the points of zero total moment on that curve. Thus:

* See Art. 2 of a paper by the author *On ruled surfaces*, recently communicated to The Mathematical Gazette.

The locus of points at which the two directions of zero moment in the normal plane are at right angles is the surface of normality, or surface of zero total moment. It is given by the equation

$$(13) \quad \mathbf{t} \cdot \text{rot } \mathbf{t} = 0.$$

Corresponding to the limit surface we next consider the locus of points at which the two directions of zero moment in the normal plane are coincident. At such a point the normal plane is tangent to the cone of Malus. The condition for the coincidence of the two directions may be found algebraically by the use of oblique curvilinear coördinates. The method is the same as that adopted in the case of the limit surface in our previous paper* already referred to. As, however, the details of the analysis are rather long, we shall here give only the final result, which may be expressed in invariant form as follows:

The locus of points at which the two directions of zero moment in the normal plane are coincident is the surface given by

$$(14) \quad 2 \operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) - (\operatorname{div} \mathbf{t})^2 + (\mathbf{t} \cdot \text{rot } \mathbf{t})^2 = 0.$$

For convenience we shall refer to this surface as the *ultimate surface*—a name suggested by the term limit surface applied in the corresponding case when the two directions of zero tendency are coincident. We may notice what the theorem becomes in the case of a normal congruence. If J and K are the mean† and Gaussian curvatures of the orthogonal surface, the equation (14) expresses that

$$4K - J^2 = 0,$$

that is to say, *the ultimate surface of a normal congruence is the locus of the umbilical points of the surfaces orthogonal to the congruence*. The interpretation is that at such points the cone of Malus consists of two planes, one of which is perpendicular to $\mathbf{t}$. Any direction in this plane is one of zero moment.

5. AXES OF NORMAL SECTIONS

We shall next prove the important property that the axes of the conic (4) bisect the angles between those of the conic (11), and vice versa. Take rectangular axes of Cartesian coördinates x, y, z so that the first is in the direction of the tangent $\mathbf{t}$ at the point P . Then if i, j, k are unit vectors in the directions of these axes, i is parallel to $\mathbf{t}$, while j and k are perpendicular

* *On congruences of curves*, Art. 4.

† *Differential Geometry*, p. 226 and p. 261.

to t and to each other. The derivatives of t in the directions of the coördinate axes may be expressed in the form

$$a j + a' k, \quad b j + b' k, \quad c j + c' k$$

respectively. Then since $r = yj + zk$, the conic (4) is

$$x = 0, \quad b y^2 + (b' + c) yz + c' z^2 = 1,$$

and the axes of the conic, being the bisectors of the angles between its asymptotes, are given by

$$(A) \quad x = 0, \quad \frac{y^2 - z^2}{b - c'} = \frac{2yz}{b' + c}.$$

Similarly the conic (11) has for its equations

$$x = 0, \quad b' y^2 + (c' - b) yz - c z^2 = 1,$$

and its axes are given by

$$(B) \quad x = 0, \quad \frac{y^2 - z^2}{b' + c} = \frac{2yz}{c' - b}.$$

It is clear that the lines (B) are the bisectors of the angles between those given by (A), and vice versa. Hence the theorem:

The axes of the section of either of the quadrics (2) or (9) by the normal plane bisect the angles between the axes of the section of the other.

This includes as a particular case the theorem for a rectilinear congruence, that the bisectors of the angles between the focal planes are also the bisectors of the angles between the principal planes.* For the cone of Malus is the same at all points of a given ray, consisting of the two focal planes. Thus the planes bisecting the angles between the focal planes contain the axes of the conic (11) for all points of the ray. Similarly it follows from Hamilton's formula† that the principal planes bisect the angles between the asymptotes of the conic (4), and therefore contain the axes of that conic for all points of the ray. From the above theorem it therefore follows that the principal planes are inclined at an angle $\pi/4$ to the bisectors of the angles between the focal planes.

* *Differential Geometry*, p. 191.

† *Differential Geometry*, p. 189.

6. RATES OF ROTATION

Consider next the arc-rate at which a direction of zero tendency in the normal plane turns about the tangent as the point moves along the curve. If $\mathbf{a}$ is the unit vector in this direction of zero tendency we have identically

$$(15) \quad \mathbf{a} \cdot \nabla \mathbf{t} \cdot \mathbf{a} = 0.$$

Let d/ds denote differentiation along the curve. Then, if $\mathbf{b}$ is the unit vector $\mathbf{t} \times \mathbf{a}$, we may write

$$\frac{d\mathbf{a}}{ds} = \omega \mathbf{b} + \psi \mathbf{t}$$

where ω is the arc-rate of turning about the tangent. Hence on differentiating (15) we have, since $\mathbf{t}$ is perpendicular to its derivatives,

$$(\omega \mathbf{b} + \psi \mathbf{t}) \cdot \nabla \mathbf{t} \cdot \mathbf{a} + \mathbf{a} \cdot \frac{d}{ds}(\nabla \mathbf{t}) \cdot \mathbf{a} + \omega \mathbf{a} \cdot \nabla \mathbf{t} \cdot \mathbf{b} = 0,$$

which may be written

$$(16) \quad \omega [\mathbf{b} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a}] = \mathbf{a} \cdot \frac{d}{ds}(\nabla \mathbf{t}) \cdot \mathbf{a} + \psi \mathbf{t} \cdot \nabla \mathbf{t} \cdot \mathbf{a}.$$

At the limit surface of the congruence the coefficient of ω is zero. For $\mathbf{a}$ and $\mathbf{b}$ have then the directions of the axes of the conic (4), and are therefore inclined at equal angles $\pi/4$ to each of the axes of the conic (11). It follows from (12) that the moments in these directions are equal, showing that the coefficient of ω vanishes at the limit surface. To interpret this, we observe that, just as ω is the arc-rate of rotation of the direction of zero tendency in the normal plane as the point P moves along the curve, so its reciprocal is the rate at which the point P moves along the curve for rotation of the normal direction of zero tendency. Thus, since the coefficient of ω in (16) vanishes at a limit point, while in general the second member of (16) does not, we have the following theorem:

At the limit points of a curve the feet of the normals giving the directions of zero tendency in the normal plane are stationary for variation of these directions.

This is substantially the theorem found in our earlier paper by a different method, and stated in terms of common normals to the curve and nearby curves.*

Similarly we may examine the rate of rotation of a direction of zero

* On congruences of curves, Art. 4.

moment in the normal plane. If $\mathbf{a}$ is now the unit vector in such a direction we have identically

$$\mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} = 0.$$

Hence, with the same notation as before, we have on differentiation

$$(\omega \mathbf{b} + \psi \mathbf{t}) \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} + \mathbf{a} \cdot \frac{d}{ds} (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} + \omega \mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{b} = 0,$$

which may be expressed as

$$(17) \quad \omega [\mathbf{a} \cdot \nabla \mathbf{t} \cdot \mathbf{a} - \mathbf{b} \cdot \nabla \mathbf{t} \cdot \mathbf{b}] = \mathbf{a} \cdot \frac{d}{ds} (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} + \psi \mathbf{t} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a}.$$

At the ultimate surface the coefficient of ω is zero. For $\mathbf{a}$ and $\mathbf{b}$ have then the directions of the axes of the conic (11), and are therefore inclined at equal angles $\pi/4$ to those of the conic (4). It follows that the tendencies in these directions are equal, so that the coefficient of ω in (17) vanishes. In general the second member of (17) is not zero, and we have the following theorem:

At the ultimate points of a curve the feet of the normals giving the directions of zero moment in the normal plane are stationary for variation of these directions.

7. NORMAL CONGRUENCE

It is known that, for a normal congruence of straight lines, the foci coincide with the limits.* In order to extend the theorem to a normal congruence of curves, we shall make use of the system of oblique curvilinear coördinates adopted in our earlier paper. Let s be the distance measured along a curve from a given surface, called the *director surface*, while u, v are current parameters for a point on that surface. Any curve is determined by the values of u, v for the point at which it crosses the director surface. Let $\mathbf{r}$ be the position vector of a point in space, and suffixes 1, 2, 3 denote differentiation with respect to u, v, s respectively. Then the unit tangent $\mathbf{t}$ is $\mathbf{r}_3$.

Consider a point on the curve (u, v) whose distance from a point on a nearby curve $(u+du, v+dv)$ is of the second or higher order. Then, if $\mathbf{r}(u, v, s)$ and $\mathbf{r}(u+du, v+dv, s+ds)$ are these points on the two curves, we have to the first order

$$\mathbf{r}(u, v, s) = \mathbf{r}(u + du, v + dv, s + ds)$$

so that

$$(18) \quad \mathbf{r}_1 du + \mathbf{r}_2 dv + \mathbf{r}_3 ds = 0,$$

* *Differential Geometry*, Art. 100.

showing that $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are coplanar, and leading to the equation of the focal surface $[\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3] = 0$.

Since $\mathbf{t}$ is perpendicular to its derivatives, it follows on forming the scalar product of the members of (18) with $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ in turn that the quotient du/dv has the value

$$\frac{du}{dv} = -\frac{\mathbf{r}_2 \cdot \mathbf{t}_1}{\mathbf{r}_1 \cdot \mathbf{t}_1} = -\frac{\mathbf{r}_2 \cdot \mathbf{t}_2}{\mathbf{r}_1 \cdot \mathbf{t}_2} = -\frac{\mathbf{r}_2 \cdot \mathbf{t}_3}{\mathbf{r}_1 \cdot \mathbf{t}_3},$$

and consequently, for points on the focal surface,

$$(19) \quad \begin{aligned} (\mathbf{r}_2 \cdot \mathbf{t}_1)(\mathbf{r}_1 \cdot \mathbf{t}_3) &= (\mathbf{r}_1 \cdot \mathbf{t}_1)(\mathbf{r}_2 \cdot \mathbf{t}_3), \\ (\mathbf{r}_2 \cdot \mathbf{t}_2)(\mathbf{r}_1 \cdot \mathbf{t}_3) &= (\mathbf{r}_1 \cdot \mathbf{t}_2)(\mathbf{r}_2 \cdot \mathbf{t}_3). \end{aligned}$$

In terms of the magnitudes

$$\begin{aligned} a &= \mathbf{r}_1^2, & b &= \mathbf{r}_2^2, \\ f &= \mathbf{r}_2 \cdot \mathbf{r}_3, & g &= \mathbf{r}_3 \cdot \mathbf{r}_1, & h &= \mathbf{r}_1 \cdot \mathbf{r}_2 \end{aligned}$$

the first of (19) may be expressed by

$$(20) \quad (f_1 - g_2 + h_3)g_3 = a_3f_3.$$

Now for a normal congruence $\mathbf{t} \cdot \text{rot } \mathbf{t}$ vanishes identically. This may be expressed* by

$$f_1 - g_2 = gf_3 - fg_3.$$

Substituting this value in (20) we have

$$(gf_3 - fg_3 + h_3)g_3 = a_3f_3,$$

or

$$(21) \quad f_3B_3 + g_3H_3 = 0,$$

A, B, H being the cofactors of a, b, h respectively in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & 1 \end{vmatrix}.$$

Similarly from the second of (19) we find

$$(22) \quad g_3A_3 + f_3H_3 = 0.$$

From this equation and (21) it then follows that

$$(23) \quad A_3B_3 = H_3^2,$$

* On congruences of curves, Art. 8.

which is the equation of the limit surface* of the congruence. Thus at the focal points of a curve the equation of the limit surface is satisfied, and we have the following theorem:

For a normal congruence of curves the foci lie on the limit surface.

8. WHERE THE CONES ARE PAIRS OF PLANES

We have seen that the cone of Malus consists of a pair of planes at all points of a rectilinear congruence, and also at points on the ultimate surface of a normal curvilinear congruence. Let us consider at what points either of the two cones thus becomes a pair of planes. Since the tangent at the point is a generator of each cone, it is clear that when the cone consists of a pair of planes, one of these must pass through the tangent.

First consider the cone of zero tendency, and suppose that this is a pair of planes. Let $\mathbf{a}$ be the unit vector parallel to the intersection of the plane through the tangent with the normal plane at P . Then any direction at P in the plane of $\mathbf{t}$ and $\mathbf{a}$ is one of zero tendency; so that, for all values of ϕ ,

$$(\mathbf{t} \cos \phi + \mathbf{a} \sin \phi) \cdot \nabla \mathbf{t} \cdot (\mathbf{t} \cos \phi + \mathbf{a} \sin \phi) = 0.$$

Since $\mathbf{t}$ is a unit vector it is perpendicular to its derivatives. Also $\mathbf{a}$ is by hypothesis a direction of zero tendency, so that the above condition requires that

$$(24) \quad \mathbf{t} \cdot \nabla \mathbf{t} \cdot \mathbf{a} = 0,$$

which may be expressed as

$$\kappa \mathbf{n} \cdot \mathbf{a} = 0,$$

where κ is the curvature of the curve, and $\mathbf{n}$ the unit principal normal. This equation is satisfied if κ is zero, or if $\mathbf{a}$ is the unit binormal to the curve. Hence:

At points where the curvature of a curve is zero, or where the binormal is a direction of zero tendency, the cone of zero tendency consists of a pair of planes.

The former condition is satisfied at all points for a rectilinear congruence; and in this case the cone of zero tendency at any point of a ray is a pair of planes equally inclined to each principal plane.

Next consider the cone of Malus. In order that this may consist of a pair of planes, we must have, for all values of ϕ ,

$$(\mathbf{t} \cos \phi + \mathbf{a} \sin \phi) \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot (\mathbf{t} \cos \phi + \mathbf{a} \sin \phi) = 0,$$

* On congruences of curves, Art. 4.

$\mathbf{a}$ being the unit vector in the normal plane and in the plane of zero moment through the tangent. This requires that

$$(25) \quad \mathbf{t} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} = 0,$$

which may be expressed as

$$\kappa \mathbf{n} \times \mathbf{t} \cdot \mathbf{a} = 0.$$

Now $\mathbf{t} \times \mathbf{n}$ is the unit binormal to the curve. Hence, in order that (25) may hold, either κ must vanish, or $\mathbf{a}$ must be parallel to $\mathbf{n}$. Thus:

At points where the curvature of a curve is zero, or where the principal normal is a direction of zero moment, the cone of Malus consists of a pair of planes.

The former condition is satisfied at all points for a rectilinear congruence; and in this case the cone of Malus consists of the two focal planes, which are the same for all points of a given ray.

9. ISOTROPIC CONGRUENCE

The conception of an isotropic congruence* of straight lines may be generalised so as to apply to curvilinear congruences. An isotropic rectilinear congruence is one whose limit surface coincides with its surface of striction. Now at points on the former surface the normal plane is tangent to the cone of zero tendency, while at points on the latter the sum of the tendencies in two perpendicular directions in the normal plane is zero. In order that both properties may be possessed simultaneously, the cone of zero tendency must consist of two planes, one of which is the normal plane. Conversely, points at which the cone (3) behaves in this manner must lie on both the limit surface and the surface of striction. We shall therefore define an *isotropic curvilinear congruence* as one for which the normal plane is part of the cone of zero tendency at all points of the surface of striction. This surface may then be described as a limit-striction surface. At points on it the equation (7) holds simultaneously with $\text{div } \mathbf{t} = 0$.

Let $\mathbf{a}$ and $\mathbf{b}$ be a pair of perpendicular unit vectors in the normal plane. Then, at points where the above property holds, the direction of the unit vector $\mathbf{a} \cos \phi + \mathbf{b} \sin \phi$ must be one of zero tendency for all values of ϕ . Hence the equation

$$(\mathbf{a} \cos \phi + \mathbf{b} \sin \phi) \cdot \nabla \mathbf{t} \cdot (\mathbf{a} \cos \phi + \mathbf{b} \sin \phi) = 0$$

is equivalent to the three relations

* *Differential Geometry*, Art. 102.

$$\mathbf{a} \cdot \nabla \mathbf{t} \cdot \mathbf{a} = 0, \quad \mathbf{b} \cdot \nabla \mathbf{t} \cdot \mathbf{b} = 0,$$

and

$$\mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} - \mathbf{b} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{b} = 0.$$

From the last of these it follows that the moments in the directions of $\mathbf{a}$ and $\mathbf{b}$ are equal. Since this must hold for all pairs of perpendicular directions in the normal plane, the conic (11) must be a circle, real or imaginary, and the moment must be the same for all directions in that plane, having the value $\frac{1}{2} \mathbf{t} \cdot \text{rot } \mathbf{t}$. Thus at a point P on this limit-striction surface, the moment of the family of curves on any surface of the congruence has this same value.* The directions of zero moment in the normal plane are therefore minimal. Also in the particular case of a rectilinear congruence, the Gaussian curvature K has the same value at P for all the ruled surfaces of the congruence, being equal† to $-\frac{1}{4}(\mathbf{t} \cdot \text{rot } \mathbf{t})^2$, and the parameter of distribution of the ray is the same for all such surfaces.‡

For an isotropic congruence of curves the intersection of the limit-striction surface with any surface of the congruence is the line of striction§ of the family of curves on that surface; for it is the locus of points at which the tendency is zero for the direction perpendicular to the curve.

Similarly we might imagine a curvilinear congruence with the property that, at all points of the surface of normality, the cone of Malus consists of a pair of planes, one of which is the normal plane to the curve. Such points must lie also on the ultimate surface; and the coalescence of these two surfaces gives what may be called the ultimate-normality surface. As before, let $\mathbf{a}$ and $\mathbf{b}$ be perpendicular unit vectors in the normal plane. Then if this plane is part of the cone of zero moment, we must have

$$(\mathbf{a} \cos \phi + \mathbf{b} \sin \phi) \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot (\mathbf{a} \cos \phi + \mathbf{b} \sin \phi) = 0$$

for all values of ϕ . This is equivalent to

$$\mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} = 0, \quad \mathbf{b} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{b} = 0,$$

and

$$\mathbf{b} \cdot \nabla \mathbf{t} \cdot \mathbf{b} - \mathbf{a} \cdot \nabla \mathbf{t} \cdot \mathbf{a} = 0.$$

From the last of these it follows that the tendencies in the directions of $\mathbf{a}$ and $\mathbf{b}$ are equal; and since this is true for all pairs of perpendicular vectors, the conic (4) must be a circle, real or imaginary. Thus the tendency has the same value, $\frac{1}{2} \text{div } \mathbf{t}$, for all directions in the normal plane, being also equal

* *Differential Geometry*, Art. 130.

† *On ruled surfaces*, Art. 2.

‡ *Differential Geometry*, p. 198.

§ *Ibid.*, p. 249.

to the divergence* at that point of the family of curves on any surface of the congruence through the point.

For a congruence of this nature the directions of zero tendency in the normal plane are minimal at points on the ultimate-normality surface. Also the intersection of this surface with any surface of the congruence is a line of zero moment for the family of curves on the latter,† for the moment of this family is zero for the direction in the surface perpendicular to the curve.

10. THREE ORTHOGONAL CONGRUENCES

Finally let us consider briefly some properties of three curvilinear congruences cutting one another orthogonally at all points. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the unit tangents to the curves at any point, so that

$$\mathbf{a} = \mathbf{b} \times \mathbf{c}, \quad \mathbf{b} = \mathbf{c} \times \mathbf{a}, \quad \mathbf{c} = \mathbf{a} \times \mathbf{b}.$$

The total moments A , B , C of the three congruences have the values

$$A = \mathbf{a} \cdot \text{rot } \mathbf{a}, \quad B = \mathbf{b} \cdot \text{rot } \mathbf{b}, \quad C = \mathbf{c} \cdot \text{rot } \mathbf{c}.$$

From the last equation we have

$$C = \mathbf{c} \cdot \text{rot } (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \cdot (\mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b} + \mathbf{a} \text{ div } \mathbf{b} - \mathbf{b} \text{ div } \mathbf{a})$$

or

$$(26) \quad C = \mathbf{b} \cdot (\nabla \mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} + \mathbf{a} \cdot (\nabla \mathbf{b} \times \mathbf{b}) \cdot \mathbf{a}.$$

Also since $\mathbf{a}$ is perpendicular to $\mathbf{b}$ it follows that

$$0 = \nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{a} + \mathbf{a} \times \text{rot } \mathbf{b} + \mathbf{b} \times \text{rot } \mathbf{a},$$

and consequently, on forming the scalar product with $\mathbf{b} \times \mathbf{a}$, we have

$$(27) \quad 0 = \mathbf{a} \cdot (\nabla \mathbf{b} \times \mathbf{b}) \cdot \mathbf{a} - \mathbf{b} \cdot (\nabla \mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} - B + A.$$

From this equation and (26) it follows that

$$(28) \quad \begin{aligned} \mathbf{a} \cdot (\nabla \mathbf{b} \times \mathbf{b}) \cdot \mathbf{a} &= \frac{1}{2}(B + C - A), \\ \mathbf{b} \cdot (\nabla \mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} &= \frac{1}{2}(A - B + C). \end{aligned}$$

Similarly we may show that

$$(29) \quad \begin{aligned} \mathbf{a} \cdot (\nabla \mathbf{c} \times \mathbf{c}) \cdot \mathbf{a} &= \frac{1}{2}(B + C - A), \\ \mathbf{c} \cdot (\nabla \mathbf{a} \times \mathbf{a}) \cdot \mathbf{c} &= \frac{1}{2}(A + B - C). \end{aligned}$$

From (28) and (29) it follows that the moments of the second and third congruences in the direction of the first have the same value. If we write

$$2S = A + B + C$$

we may state the above results as follows:

* *Differential Geometry*, p. 259 and Art. 116.

† *Ibid.*, p. 258.

The moments of the second and third congruences in the direction of the first have each the value $S - A$; those of the first and third in the direction of the second have the value $S - B$, and those of the first and second in the direction of the third the value $S - C$.

Thus the moments of the three orthogonal congruences are not independent, but are connected by the relations expressed in this theorem.

An important particular case is that in which two of the congruences are normal, say the first and second. Then A and B both vanish; and the curves of the third congruence are the lines of intersection of two orthogonal families of surfaces, whose unit normals are $\mathbf{a}$ and $\mathbf{b}$ respectively. The moment of the third congruence in either of the directions $\mathbf{a}$ or $\mathbf{b}$ is equal to $\frac{1}{2} C$; that is to say:

If the curves of a congruence are the lines of intersection of two orthogonal families of surfaces, the moments of the congruence in the directions of the normals to these surfaces are equal to each other and to half the total moment of the congruence.

The theorem just enunciated has several known theorems as immediate corollaries. The moment of the congruence in the direction of $\mathbf{b}$ is the moment of the family of curves on the surface whose normal is $\mathbf{a}$; and this vanishes identically only when these curves are lines of curvature on the surface.* Similarly the moment of the congruence in the direction of $\mathbf{a}$ is that of the family of curves on the surface whose normal is $\mathbf{b}$. Hence the theorem that if the curves of intersection are lines of curvature on one of the orthogonal families of surfaces, they are lines of curvature also on the other.† In that case C is zero, and the curves of intersection constitute a normal congruence. Thus we have Darboux's theorem that there is a third family of surfaces orthogonal to the first two.‡ Conversely, if C is zero, the moment of the family of curves on each of the orthogonal surfaces is zero, and the curves are therefore lines of curvature. Hence we have Dupin's theorem that the curves of intersection of the surfaces of a triply orthogonal system are lines of curvature on these surfaces.§

* *Differential Geometry*, Art. 130.

† *Ibid.*, Art. 112.

‡ Darboux, loc. cit., p. 276.

§ *Differential Geometry*, Art. 109.